

PARABOLIC EQUATIONS WITH MEASURABLE COEFFICIENTS

DOYOON KIM AND N. V. KRYLOV

ABSTRACT. We investigate the unique solvability of second order parabolic equations in non-divergence form in $W_p^{1,2}((0, T) \times \mathbb{R}^d)$, $p \geq 2$. The leading coefficients are only measurable in either one spatial variable or time and one spatial variable. In addition, they are VMO (vanishing mean oscillation) with respect to the remaining variables.

1. INTRODUCTION

This paper is a natural continuation of our previous investigations [9], [8]. By combining the techniques from these articles we investigate parabolic equations of type

$$u_t + a^{jk}(t, x)u_{x^j x^k} + b^j(t, x)u_{x^j} + c(t, x)u = f \quad (1.1)$$

in Sobolev spaces $W_p^{1,2}$ with $p \geq 2$ and the coefficients being just measurable in x^1 but VMO with respect to other variables. Here

$$(t, x) \in \mathbb{R}^{d+1} = \{(t, x^1, x') : t, x^1 \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$$

and the equation is assumed to be uniformly nondegenerate with bounded coefficients.

One of the advantages of having a “good” theory for such equations is demonstrated in [8] while treating the Dirichlet and Neumann problems, the issues addressed in this paper as well.

The amazing fact that there is a solvability theory in Sobolev spaces for elliptic and parabolic equations with discontinuous but VMO coefficients was discovered in [2], [3], and [1]. Before that the Sobolev space theory was established for some other types of discontinuities [11], [10], [4], [16] (see also [6] ([7]) for a modern approach covering $p \neq 2$ in the elliptic (parabolic) case). Solvability theory for discontinuous coefficients is important not only from pure theoretical point of view but also from the point of view of applications, for instance,

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to random diffusion processes, see, for instance, [17], [9]. Observe that the class of equations with VMO coefficients and the class of equations with discontinuities treated in [11], [4], [16], [6], [7] have no common members apart from the equations with just continuous coefficients. In this paper we show that there is a unified approach to both cases allowing one to treat equations possessing one properties with respect to some variables and other properties with respect to the remaining ones. Here we show that the coefficients $a^{jk}(t, x^1, x')$ may be just measurable in x^1 and VMO in (t, x') . Even though the equations (and partly the results) of the present article and [8] are more general than those from [1], [4], [7], [16], they are not general enough to absorb [9], where equations are considered whose coefficients a^{jk} are allowed to be *measurable* in t and VMO in x . Furthermore, the results here cover those of [7] only for $p \geq 2$. On the other hand, in [9] and [7] the coefficients only measurable in x^1 are not allowed. Thus, the classes of equations here and in [9], [7] are quite different.

It is worth noting that after [2], [3], [1] there were very many publications on elliptic and parabolic equations with VMO coefficients (see, for instance, the above mentioned references and [5], [12], [13], [14], [15], and many references therein). The approach we employ here is quite different from the approaches of other authors and is taken from [9].

This paper is organized as follows. In section 2 we present our main results. The case $p = 2$ is investigated in section 3. In section 4 we present some auxiliary results which are needed for the proof of Theorem 2.5. In section 5 we prove Theorem 2.5.

A few words about notation. As is seen from the above by (t, x) we denote a point in \mathbb{R}^{d+1} , i.e., $(t, x) = (t, x^1, x') \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}$, where $t \in \mathbb{R}$, $x^1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$, and $x = (x^1, x') \in \mathbb{R}^d$. By $|u|_0$ we mean the sup norm of u over the domain where u is defined. In this paper, we write $N = N(d, \dots)$ if N is a constant depending only on d, \dots .

2. MAIN RESULTS

We consider the parabolic equation (1.1) with coefficients a^{jk} , b^j , and c satisfying the following assumption.

Assumption 2.1. The coefficients a^{jk} , b^j , and c are measurable functions defined on \mathbb{R}^{d+1} , $a^{jk} = a^{kj}$. There exist positive constants $\delta \in (0, 1)$ and K such that

$$|b^j(t, x)| \leq K, \quad |c(t, x)| \leq K,$$

$$\delta |\vartheta|^2 \leq \sum_{j,k=1}^d a^{jk}(t, x) \vartheta^j \vartheta^k \leq \delta^{-1} |\vartheta|^2$$

for any $(t, x) \in \mathbb{R}^{d+1}$ and $\vartheta \in \mathbb{R}^d$.

We look for solutions of parabolic equations in the usual Sobolev space

$$W_p^{1,2}((S, T) \times \mathbb{R}^d) = \{u : u, u_t, u_x, u_{xx} \in L_p((S, T) \times \mathbb{R}^d)\},$$

$-\infty \leq S < T \leq \infty$ with usual norm. Throughout the paper, as in [9], we set

$$\Omega_T = (0, T) \times \mathbb{R}^d.$$

Thus, for instance,

$$L_p(\Omega_T) = L_p((0, T) \times \mathbb{R}^d), \quad W_p^{1,2}(\Omega_T) = W_p^{1,2}((0, T) \times \mathbb{R}^d).$$

By $\overset{0}{W}_p^{1,2}(\Omega_T)$ we mean the collection of functions in $W_p^{1,2}(\Omega_T)$ vanishing at $t = T$. We denote the differential operator in (1.1) by L , that is,

$$Lu = u_t + a^{jk} u_{x^j x^k} + b^j u_{x^j} + cu.$$

Our first result is about the case $p = 2$. In this case, we do not require any regularity assumptions on the coefficients a^{jk} if they are functions of only $(t, x^1) \in \mathbb{R}^2$.

Theorem 2.2. *Let Assumption 2.1 be satisfied and let the coefficients a^{ij} be independent of $x' \in \mathbb{R}^{d-1}$. Then for any $f \in L_2(\Omega_T)$, there exists a unique $u \in \overset{0}{W}_2^{1,2}(\Omega_T)$ satisfying $Lu = f$ in $(0, T) \times \mathbb{R}^d$. In addition, there is a constant $N = N(d, \delta, K, T)$ such that, for any $u \in \overset{0}{W}_2^{1,2}(\Omega_T)$,*

$$\|u\|_{\overset{0}{W}_2^{1,2}(\Omega_T)} \leq N \|Lu\|_{L_2(\Omega_T)}.$$

Remark 2.3. The assertion of Theorem 2.2 is also valid if $a^{jk}(t, x)$ are uniformly continuous as functions of $x' \in \mathbb{R}^{d-1}$ uniformly in $(t, x^1) \in \mathbb{R}^2$. This can be shown by using the standard techniques based on partitions of unity and considering the equation on small time intervals allowing one to absorb the L_2 -norm into the $\overset{0}{W}_2^{1,2}$ -norm. Actually, there also is a standard way, which can be found, for instance, in [9], to avoid solving the equation step by step on small time intervals moving down from $t = T$ to $t = 0$.

If $p \in (2, \infty)$, we suppose that the coefficients a^{jk} are measurable in $x^1 \in \mathbb{R}$ and VMO in $(t, x') \in \mathbb{R}^d$. To state this assumption precisely, we introduce the following notation. Let

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\},$$

$$B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\},$$

$$Q_r(t, x) = (t, t + r^2) \times B_r(x), \quad \Gamma_r(t, x') = (t, t + r^2) \times B'_r(x'),$$

$$\Lambda_r(t, x) = (t, t + r^2) \times (x^1 - r, x^1 + r) \times B'_r(x').$$

Set $B_r = B_r(0)$, $B'_r = B'_r(0)$, $Q_r = Q_r(0)$, and so on. By $|B'_r|$ we mean the $d - 1$ -dimensional volume of $B'_r(0)$. Denote $a = (a^{jk})$ and

$$\text{osc}_{(t,x')} (a, \Lambda_r(t, x)) = r^{-5} |B'_r|^{-2} \int_{x^1-r}^{x^1+r} A_{(t,x')}(\tau) d\tau,$$

where

$$A_{(t,x')}(\tau) = \int_{(\sigma, y'), (\varrho, z') \in \Gamma_r(t, x')} |a(\sigma, \tau, y') - a(\varrho, \tau, z')| dy' dz' d\sigma d\varrho.$$

Also denote

$$a_R^\# = \sup_{(t,x) \in \mathbb{R}^{d+1}} \sup_{r \leq R} \text{osc}_{(t,x')} (a, \Lambda_r(t, x)).$$

Assumption 2.4. There is a continuous function $\omega(t)$ defined on $[0, \infty)$ such that $\omega(0) = 0$ and $a_R^\# \leq \omega(R)$ for all $R \in [0, \infty)$.

Theorem 2.5. Let $p \in (2, \infty)$ and let Assumptions 2.1 and 2.4 be satisfied. Then for any $f \in L_p(\Omega_T)$, there exists a unique $u \in \overset{0}{W}_p^{1,2}(\Omega_T)$ such that $Lu = f$ in $(0, T) \times \mathbb{R}^d$. Furthermore, there is a constant $N = N(d, \delta, K, p, \omega, T)$ such that, for any $u \in \overset{0}{W}_p^{1,2}(\Omega_T)$,

$$\|u\|_{\overset{0}{W}_p^{1,2}(\Omega_T)} \leq N \|Lu\|_{L_p(\Omega_T)}.$$

Remark 2.6. As usual in such situations, from our proofs one can see that instead of the assumption that $a_R^\# \rightarrow 0$ as $R \downarrow 0$, actually, we are using that there exists $R \in (0, \infty)$ such that $a_R^\# \leq \varepsilon$, where $\varepsilon > 0$ is a constant depending only on other parameters of the problem.

We now show how to treat the Dirichlet and oblique derivative problems for parabolic equations in half spaces. By the fact that coefficients are allowed to be measurable in one direction, in solving these problems, we need only the results for equations in the whole space. Denote

$$\begin{aligned} \mathbb{R}_+^d &= \{x \in \mathbb{R}^d : x^1 > 0\}, \quad \Omega_T^+ = (0, T) \times \mathbb{R}_+^d, \\ \partial_t \Omega_T^+ &= \{(T, x) : x \in \mathbb{R}_+^d\}, \quad \partial_x \Omega_T^+ = \{(t, 0, x') : 0 \leq t \leq T, x' \in \mathbb{R}^{d-1}\}, \\ \partial' \Omega_T^+ &= \partial_t \Omega_T^+ \cup \partial_x \Omega_T^+. \end{aligned}$$

Below in this section we suppose that coefficients a^{jk} , b^j , and c satisfy Assumption 2.1.

Theorem 2.7. Let $2 \leq p < \infty$. Assume that a^{jk} are independent of $x' \in \mathbb{R}^{d-1}$ if $p = 2$. In case $p > 2$, we assume that a^{jk} satisfy Assumption 2.4. Then for any $f \in L_p(\Omega_T^+)$, there exists a unique $u \in \overset{0}{W}_p^{1,2}(\Omega_T^+)$ such that $Lu = f$ in $(0, T) \times \mathbb{R}_+^d$ and $u = 0$ on $\partial' \Omega_T^+$.

Proof. Introduce a new operator $\hat{L}v = \hat{a}^{jk}v_{x^j x^k} + \hat{b}v_{x^j} + \hat{c}v$, where \hat{a}^{jk} , \hat{b}^j , and \hat{c} are defined as either even or odd extensions of a^{jk} , b^j , and c . Specifically, for $j = k = 1$ or $j, k \in \{2, \dots, d\}$, even extensions:

$$\hat{a}^{jk} = a^{jk}(t, x^1, x') \quad x^1 \geq 0, \quad \hat{a}^{jk} = a^{jk}(t, -x^1, x') \quad x^1 < 0.$$

For $j = 2, \dots, d$, odd extensions:

$$\hat{a}^{1j} = a^{1j}(t, x^1, x') \quad x^1 \geq 0, \quad \hat{a}^{1j} = -a^{1j}(t, -x^1, x') \quad x^1 < 0.$$

Also set $\hat{a}^{j1} = \hat{a}^{1j}$. Similarly, \hat{b}^1 is the odd extension of b^1 , and \hat{b}^j , $j = 2, \dots, d$, and \hat{c} are even extensions of b^j and c respectively. We see that the coefficients \hat{a}^{jk} , \hat{b}^j , and \hat{c} satisfy Assumption 2.1. In addition, if $p > 2$, the coefficients \hat{a}^{jk} satisfy Assumption 2.4 with 2ω .

Let \hat{f} be the odd extension of f . Then it follows by Theorem 2.2 or 2.5 that there exists a unique $u \in \overset{0}{W}_p^{1,2}(\Omega_T)$ such that $\hat{L}u = \hat{f}$. It is easy to check that $-u(t, -x^1, x') \in \overset{0}{W}_p^{1,2}(\Omega_T)$ also satisfy the same equation, so by uniqueness we have $u(t, x^1, x') = -u(t, -x^1, x')$. This and the fact that $u \in \overset{0}{W}_p^{1,2}(\Omega_T)$ show that u , as a function defined on $(0, T) \times \mathbb{R}_+^d$, is a solution to $Lu = f$ satisfying $u = 0$ on $\partial'\Omega_T^+$.

Uniqueness follows from the fact that the odd extension of a solution u belongs to $\overset{0}{W}_p^{1,2}(\Omega_T)$ and the uniqueness of solutions to equations in Ω_T . \square

The following theorem addresses oblique derivative problems.

Theorem 2.8. *Let p and a^{jk} be as in Theorem 2.7. Let $\ell = (\ell^1, \dots, \ell^d)$ be a vector in \mathbb{R}^d with $\ell^1 > 0$. Then for any $f \in L_p(\Omega_T^+)$, there exists a unique $u \in W_p^{1,2}(\Omega_T^+)$ satisfying $Lu = f$ in $(0, T) \times \mathbb{R}_+^d$, $\ell^j u_{x^j} = 0$ on $\partial_x \Omega_T^+$, and $u = 0$ on $\partial_t \Omega_T^+ = 0$.*

Proof. Let $\varphi(x) = (\ell^1 x^1, \ell' x^1 + x')$, where $\ell' = (\ell^2, \dots, \ell^d)$. Using this linear transformation and its inverse, we reduce the above problem to a problem with Neumann boundary condition on $\partial_x \Omega_T^+$. Note that, in case $p > 2$, the coefficients of the transformed equation satisfy Assumption 2.4 with $N\omega(N\cdot)$, where N depends only on d and ℓ . Then the latter problem is solved as in the proof of Theorem 2.7 with the even extension of f . \square

Remark 2.9. Solutions to problems in the above two theorems satisfy the L_p -estimate. That is, if u is a solution, then

$$\|u\|_{W_p^{1,2}(\Omega_T^+)} \leq N\|f\|_{L_p(\Omega_T^+)},$$

where N is a constant depending only on some or all parameters – d , δ , K , p , ω , T , ℓ .

3. PROOF OF THEOREM 2.2

Introduce

$$L_0 u(t, x) = u_t(t, x) + a^{jk}(t, x^1) u_{x^j x^k}(t, x), \quad (3.1)$$

Lemma 3.1. *Assume that $d = 1$. Then for any $\lambda > 0$ and $f \in L_2(\mathbb{R}^2)$ there exists a unique solution $u \in W_2^{1,2}(\mathbb{R}^2)$ of the equation $L_0 u - \lambda u = f$. Furthermore, there is a constant $N = N(\delta)$ such that for any $\lambda \geq 0$ and $u \in W_2^{1,2}(\mathbb{R}^2)$ we have*

$$\begin{aligned} & \|u_t\|_{L_2(\mathbb{R}^2)} + \|u_{xx}\|_{L_2(\mathbb{R}^2)} + \sqrt{\lambda} \|u_x\|_{L_2(\mathbb{R}^2)} \\ & + \lambda \|u\|_{L_2(\mathbb{R}^2)} \leq N \|L_0 u - \lambda u\|_{L_2(\mathbb{R}^2)}. \end{aligned} \quad (3.2)$$

Proof. As usual we only need prove (3.2) and only for $u \in C_0^\infty(\mathbb{R}^2)$. Take such a function, denote $a = a^{11}$, $f := L_0 u - \lambda u$, and write

$$a^{-1/2} f = a^{1/2} u_{xx} + a^{-1/2} (u_t - \lambda u),$$

$$a^{-1} f^2 = a u_{xx}^2 + 2u_{xx}(u_t - \lambda u) + a^{-1} (u_t - \lambda u)^2.$$

Then integrate through the last equation over \mathbb{R}^2 and notice that

$$2 \int_{\mathbb{R}^2} u_{xx} u_t \, dx dt = -2 \int_{\mathbb{R}^2} u_x u_{xt} \, dx dt = - \int_{\mathbb{R}^2} \frac{\partial}{\partial t} u_x^2 \, dt dx = 0,$$

$$2 \int_{\mathbb{R}^2} u_{xx} u \, dx dt = - \int_{\mathbb{R}^2} u_x^2 \, dx dt.$$

Then we find

$$\delta^{-1} \int_{\mathbb{R}^2} f^2 \, dx dt \geq \delta \int_{\mathbb{R}^2} u_{xx}^2 \, dx dt + \lambda \int_{\mathbb{R}^2} u_x^2 \, dx dt + \delta \int_{\mathbb{R}^2} (u_t - \lambda u)^2 \, dx dt.$$

Upon observing that

$$\int_{\mathbb{R}^2} (u_t - \lambda u)^2 \, dx dt = \int_{\mathbb{R}^2} u_t^2 \, dx dt - 2\lambda \int_{\mathbb{R}^2} u_t u \, dx dt + \lambda^2 \int_{\mathbb{R}^2} u^2 \, dx dt,$$

$$2 \int_{\mathbb{R}^2} u_t u \, dx dt = - \int_{\mathbb{R}^2} \frac{\partial}{\partial t} u^2 \, dx dt = 0$$

we finish the proof. \square

We now generalize Lemma 3.1 to cover the multidimensional case.

Theorem 3.2. *For any $\lambda > 0$ and $f \in L_2(\mathbb{R}^{d+1})$ there exists a unique solution $u \in W_2^{1,2}(\mathbb{R}^{d+1})$ of the equation $L_0 u - \lambda u = f$. Furthermore, there is a constant $N = N(\delta)$ such that for any $\lambda \geq 0$ and $u \in W_2^{1,2}(\mathbb{R}^{d+1})$ we have*

$$\|u_t\|_{L_2(\mathbb{R}^{d+1})} + \|u_{xx}\|_{L_2(\mathbb{R}^{d+1})} + \sqrt{\lambda} \|u_x\|_{L_2(\mathbb{R}^{d+1})}$$

$$+\lambda\|u\|_{L_2(\mathbb{R}^{d+1})} \leq N\|L_0u - \lambda u\|_{L_2(\mathbb{R}^{d+1})}. \quad (3.3)$$

It is worth saying that by

$$\|u_x\|_{L_2(\mathbb{R}^{d+1})} \quad \text{and} \quad \|u_{xx}\|_{L_2(\mathbb{R}^{d+1})}$$

in (3.3) we mean L_2 -norms of

$$\left(\sum_k |u_{x^k}|^2\right)^{1/2} \quad \text{and} \quad \left(\sum_{k,j} |u_{x^k x^j}|^2\right)^{1/2},$$

respectively. Different definitions could make N depend also on d .

We prove this theorem after some preparations. Again it suffices to only prove (3.3) and only for $u \in C_0^\infty(\mathbb{R}^{d+1})$. In addition we may assume that a^{ij} are infinitely differentiable. Fix such u , a^{ij} , and $\lambda \geq 0$ and set

$$f := L_0u - \lambda u.$$

Let $\xi \in \mathbb{R}^{d-1}$ and let $\tilde{\psi}(t, x^1, \xi)$ denote the Fourier transform of $\psi(t, x^1, x')$ with respect to $x' \in \mathbb{R}^{d-1}$. By taking the Fourier transform (with respect to $x' \in \mathbb{R}^{d-1}$), we obtain

$$\tilde{u}_t(t, x^1, \xi) + \mathbf{a}(t, x^1)\tilde{u}_{x^1 x^1}(t, x^1, \xi) + i 2 \mathbf{b}(t, x^1, \xi)\tilde{u}_{x^1}(t, x^1, \xi)$$

$$- \mathbf{c}(t, x^1, \xi)\tilde{u}(t, x^1, \xi) - \lambda\tilde{u}(t, x^1, \xi) = \tilde{f}(t, x^1, \xi), \quad (3.4)$$

where $i = \sqrt{-1}$,

$$\mathbf{a}(x^1) = a^{11}(x^1), \quad \mathbf{b}(t, x^1, \xi) = \sum_{j=2}^d a^{1j}(t, x^1)\xi^j,$$

$$\mathbf{c}(t, x^1, \xi) = \sum_{j,k=2}^d a^{jk}(t, x^1)\xi^j\xi^k.$$

Introduce a function

$$\rho(t, x^1, \xi) = \tilde{u}(t, x^1, \xi) e^{i\phi(t, x^1, \xi)},$$

where $\phi(t, 0, \xi) = 0$ and $\phi_{x^1}(t, x^1, \xi) = \mathbf{a}^{-1}\mathbf{b}(t, x^1, \xi)$. It is easy to see that ρ satisfies

$$\rho_t + \mathbf{a}\rho_{x^1 x^1} - (\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 + \lambda + i\phi_t + i\mathbf{a}\phi_{x^1 x^1})\rho = \tilde{f}e^{i\phi}. \quad (3.5)$$

In the following lemma ξ is considered as a parameter.

Lemma 3.3. *Let $|\xi|^2 + \lambda > 0$. Then we have*

$$|\rho(t, x^1, \xi)| \leq \hat{\rho}(t, x^1, \xi), \quad (3.6)$$

where, for each $\xi \in \mathbb{R}^{d-1}$, $\hat{\rho}(t, x^1, \xi)$ is the unique $W_2^{1,2}(\mathbb{R}^2)$ solution of

$$\hat{\rho}_t + \mathbf{a}\hat{\rho}_{x^1x^1} - (\lambda + \delta^3|\xi|^2)\hat{\rho} = -|\tilde{f}|. \quad (3.7)$$

In particular, (by Lemma 3.1)

$$\begin{aligned} (|\xi|^2 + \lambda)\|\tilde{u}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)} &= (|\xi|^2 + \lambda)\|\rho(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)} \\ &\leq (|\xi|^2 + \lambda)\|\hat{\rho}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)} \leq N(\delta)\|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}. \end{aligned} \quad (3.8)$$

Proof. First, observe that by Lemma 3.1 the function $\hat{\rho}$ indeed exists and by the maximum principle it is nonnegative. Also since $|\tilde{f}|$ is Lipschitz continuous, $\hat{\rho}$ is twice continuously differentiable in x and once in t .

Assume that, for a fixed ξ , (3.6) is violated. Then, due to the fact that ρ has a compact support, there is a point (t_0, x_0^1) such that

$$|\rho(t_0, x_0^1)| - \hat{\rho}(t_0, x_0^1) = \max_{\mathbb{R}^2}(|\rho(t, x^1)| - \hat{\rho}(t, x^1)) > 0. \quad (3.9)$$

Since $|\rho(t_0, x_0^1)| > 0$ and ρ is smooth, the function $|\rho|$ is twice differentiable at (t_0, x_0) and at this point

$$\begin{aligned} |\rho|_{x^1} &= \frac{\Re(\bar{\rho}\rho_{x^1})}{|\rho|} = \hat{\rho}_{x^1}, \quad |\rho|_t = \frac{\Re(\bar{\rho}\rho_t)}{|\rho|} = \hat{\rho}_t, \\ |\rho|_{x^1x^1} &= \frac{1}{|\rho|^3}(|\rho|^2|\rho_{x^1}|^2 - (\Re(\bar{\rho}\rho_{x^1}))^2) + \frac{1}{|\rho|}\Re(\bar{\rho}\rho_{x^1x^1}) \leq \hat{\rho}_{x^1x^1}. \end{aligned}$$

Obviously, $(\Re(\bar{\rho}\rho_{x^1}))^2 \leq |\rho|^2|\rho_{x^1}|^2$, so that we also have

$$\frac{1}{|\rho|}\Re(\bar{\rho}\rho_{x^1x^1}) \leq \hat{\rho}_{x^1x^1}.$$

Next, we multiply (3.5) by $\eta := \bar{\rho}/|\rho|$ and take real parts of both sides to get

$$\Re(\eta\tilde{f}e^{i\phi}) = \Re(\eta\rho_t) + \mathbf{a}\Re(\eta\rho_{x^1x^1}) - (\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 + \lambda)|\rho| \quad (3.10)$$

Concentrate on this equation at the point (t_0, x_0) and use the above manipulations with the derivatives to see that at (t_0, x_0)

$$\begin{aligned} \Re(\eta\tilde{f}e^{i\phi}) &\leq \hat{\rho}_t + \mathbf{a}\hat{\rho}_{x^1x^1} - (\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 + \lambda)|\rho| \\ &= -|\tilde{f}| + (\delta^3|\xi|^2 + \lambda)\hat{\rho} - (\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 + \lambda)|\rho|. \end{aligned}$$

Here, $|\rho| > \hat{\rho} \geq 0$ and as is easy to check (see, for instance, Lemma 3.1 in [8]), $\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 \geq \delta^3|\xi|^2$. Therefore, (always at (t_0, x_0))

$$(\mathbf{c} - \mathbf{a}^{-1}\mathbf{b}^2 + \lambda)|\rho| > (\delta^3|\xi|^2 + \lambda)\hat{\rho},$$

so that we get

$$\Re(\eta \tilde{f} e^{i\phi}) \leq -|\tilde{f}| + (\delta^3 |\xi|^2 + \lambda) \hat{\rho} - (\mathbf{c} - \mathbf{a}^{-1} \mathbf{b}^2 + \lambda) |\rho| < -|\tilde{f}|.$$

This leads to a contradiction because $|\eta| = 1$ and proves the lemma. \square

Lemma 3.4. *For any $\varepsilon > 0$, there exists a constant $N(\varepsilon, \delta)$ such that*

$$(|\xi| + \sqrt{\lambda}) \|\rho_x(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)} \leq N(\varepsilon, \delta) \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)} + \varepsilon \|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}. \quad (3.11)$$

Proof. We go back to equation (3.10), which we multiply by $|\rho|$, divide by \mathbf{a} , then integrate over \mathbb{R}^2 , and use that $\mathbf{c} \leq \delta^{-1} |\xi|^2$ and $|b| \leq \delta^{-1} |\xi|$. We also use the fact that

$$\begin{aligned} \Re \int_{\mathbb{R}^2} \bar{\rho} \rho_{x^1 x^1} dx dt &= - \int_{\mathbb{R}^2} |\rho_{x^1}|^2 dx dt, \\ 2\Re(\bar{\rho} \rho_t) &= \frac{\partial}{\partial t} |\rho|^2 = \frac{\partial}{\partial t} |\tilde{u}|^2 = 2\Re(\tilde{u} \tilde{u}_t). \end{aligned}$$

Then we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\rho_{x^1}|^2 dx dt &\leq 2 \int_{\mathbb{R}^2} \mathbf{a}^{-1} |\tilde{u} \tilde{u}_t| dx dt \\ &+ N(\lambda + |\xi|^2) \int_{\mathbb{R}^2} |\rho|^2 dx dt + 2 \int_{\mathbb{R}^2} \mathbf{a}^{-1} |\rho \tilde{f}| dx dt. \end{aligned}$$

We estimate the terms on the right by using Young's inequality and assuming without losing generality that $\lambda + |\xi|^2 \neq 0$. For instance,

$$\begin{aligned} 2 \int_{\mathbb{R}^2} \mathbf{a}^{-1} |\tilde{u} \tilde{u}_t| dx dt &\leq \delta^{-2} \varepsilon^{-1} (\lambda + |\xi|^2) \int_{\mathbb{R}^2} |\tilde{u}|^2 dx dt \\ &+ \varepsilon (\lambda + |\xi|^2)^{-1} \int_{\mathbb{R}^2} |\tilde{u}_t|^2 dx dt. \end{aligned}$$

We also use (3.8). Then we easily get (3.11). \square

Proof of Theorem 3.2. Since

$$\tilde{u} = \rho e^{-i\phi}, \quad \tilde{u}_{x^1} = [\rho_{x^1} - i \mathbf{a}^{-1} \mathbf{b} \rho] e^{-i\phi},$$

and $|\mathbf{b}| \leq N|\xi|$, Lemmas 3.3 and 3.4 imply that for any $\varepsilon > 0$ there is an $N(\varepsilon, \delta)$ such that

$$\begin{aligned} &(|\xi|^4 + \lambda^2) \|\tilde{u}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + (|\xi|^2 + \lambda) \|\tilde{u}_{x^1}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 \\ &\leq N(\varepsilon, \delta) \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + \varepsilon \|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2. \end{aligned} \quad (3.12)$$

Then (3.4) shows that for any $\varepsilon > 0$ there is an $N(\varepsilon, \delta)$ such that

$$\|(\tilde{u}_t + \mathbf{a} \tilde{u}_{x^1 x^1})(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 \leq N(\varepsilon, \delta) \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + \varepsilon \|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2,$$

which after being combined with Lemma 3.1 (with $\lambda = 0$ there) leads to

$$\begin{aligned} & \|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + \|\tilde{u}_{x^1 x^1}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 \\ & \leq N(\varepsilon, \delta) \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + \varepsilon \|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2. \end{aligned} \quad (3.13)$$

The reader might have noticed that in the above computations the constants $N(\varepsilon, \delta)$ are changing from line to line and ε was sometimes multiplied by a constant of type $N(\delta)$. However, $N(\delta)\varepsilon$ is as arbitrary as ε . Upon taking $\varepsilon = 1/2$ in (3.13) we conclude that

$$\|\tilde{u}_t(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + \|\tilde{u}_{x^1 x^1}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 \leq N \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2. \quad (3.14)$$

After that (3.12) yields

$$\begin{aligned} & (|\xi|^4 + \lambda^2) \|\tilde{u}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 + (|\xi|^2 + \lambda) \|\tilde{u}_{x^1}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2 \\ & \leq N \|\tilde{f}(\cdot, \cdot, \xi)\|_{L_2(\mathbb{R}^2)}^2. \end{aligned} \quad (3.15)$$

To get (3.3) now it only remains to integrate through (3.14) and (3.15) with respect to ξ and use Parseval's identity. The theorem is proved. \square

Theorem 2.2 is derived from Theorem 3.2 in a standard way, which can be found, for instance, in [9]. Theorem 2.2 is proved.

4. AUXILIARY RESULTS FOR EQUATION IN $W_p^{1,2}(\mathbb{R}^{d+1})$

We assume in this section that a^{jk} are measurable functions only of $x^1 \in \mathbb{R}$. Set

$$L_0 u(t, x) = u_t(t, x) + a^{jk}(x^1) u_{x^j x^k}(t, x).$$

By $\partial' Q_r(t, x)$ we mean the parabolic boundary of $Q_r(t, x)$ defined as

$$\partial' Q_r(t, x) = ([t, t + r^2] \times \partial B_r(x)) \cup \{(t + r^2, y) : y \in B_r(x)\}.$$

Lemma 4.1. *There exists $N = N(d, \delta)$ such that, for $u \in W_2^{1,2}(Q_r)$ with $u|_{\partial' Q_r} = 0$, we have*

$$r^2 \int_{Q_r} |u_x|^2 dx dt + \int_{Q_r} |u|^2 dx dt \leq N r^4 \int_{Q_r} |L_0 u|^2 dx dt. \quad (4.1)$$

Proof. Assume that (4.1) is true when $r = 1$. For $u \in W_2^{1,2}(Q_r)$ with $u|_{\partial' Q_r} = 0$, we set

$$\hat{L}_0 = \frac{\partial}{\partial t} + a^{jk}(rx^1) \frac{\partial^2}{\partial x^j \partial x^k} \quad \text{and} \quad \hat{u}(t, x) = r^{-2} u(r^2 t, rx).$$

Then $\hat{u} \in W_2^{1,2}(Q_1)$ and $\hat{L}_0 \hat{u}(t, x) = L_0 u(r^2 t, rx)$ in Q_1 . Since \hat{L}_0 satisfies the same ellipticity condition as L_0 does, we have

$$\begin{aligned} \int_{Q_r} |u|^2 dx dt &= r^{d+6} \int_{Q_1} |\hat{u}|^2 dx dt \\ &\leq N r^{d+6} \int_{Q_1} |\hat{L}_0 \hat{u}|^2 dx dt = N r^4 \int_{Q_r} |L_0 u|^2 dx dt. \end{aligned}$$

Also

$$\begin{aligned} \int_{Q_r} |u_x|^2 dx dt &= r^{d+4} \int_{Q_1} |\hat{u}_x|^2 dx dt \\ &\leq N r^{d+4} \int_{Q_1} |\hat{L}_0 \hat{u}|^2 dx dt = N r^2 \int_{Q_r} |L_0 u|^2 dx dt. \end{aligned}$$

This shows that we need to prove the lemma only for $r = 1$.

In this case, we divide L_0 by $a^{11}(x^1)$. That is, by setting

$$f := u_t + a^{jk} u_{x^j x^k}, \quad \hat{a}^{jk} := a^{jk}/a^{11},$$

we have

$$u_t/a^{11} + \hat{a}^{jk} u_{x^j x^k} = f/a^{11}.$$

Then using the ellipticity of \hat{a}^{jk} and integration by parts, we obtain

$$\begin{aligned} \delta^2 \int_{Q_1} |u_x|^2 dx dt &\leq \int_{Q_1} \hat{a}^{jk} u_{x^j} u_{x^k} dx dt = - \int_{Q_1} u \hat{a}^{jk} u_{x^j x^k} dx dt \\ &= \int_{Q_1} \frac{u}{a^{11}} (u_t - f) dx dt. \end{aligned}$$

Note that

$$\int_{Q_1} \frac{u}{a^{11}} u_t dx dt = \int_{B_1} \frac{1}{a^{11}} \int_0^1 u u_t dt dx = - \int_{B_1} \frac{1}{2a^{11}} u(0, x)^2 dx \leq 0,$$

where we used the fact that a^{11} is independent of t and $u(1, x) = 0$.

Thus we have

$$\begin{aligned} \delta^2 \int_{Q_1} |u_x|^2 dx dt &\leq - \int_{Q_1} \frac{u}{a^{11}} f dx dt \\ &\leq \delta^{-1} \left(\int_{Q_1} u^2 dx dt \right)^{1/2} \left(\int_{Q_1} f^2 dx dt \right)^{1/2}. \end{aligned}$$

By using Poincaré's inequality, we estimate the integral of u^2 in the last term through that of $|u_x|^2$. This gives us the needed estimate for u_x . For the estimate for u , we use Poincaré's inequality once again. The lemma is proved. \square

Lemma 4.2. *Let $0 < r < R$. There exists $N = N(d, \delta)$ such that, for $u \in W_2^{1,2}(Q_R)$,*

$$\|u\|_{W_2^{1,2}(Q_r)} \leq N \left(\|L_0 u - u\|_{L_2(Q_R)} + (R - r)^{-2} \|u\|_{L_2(Q_R)} \right).$$

Proof. The proof is just a repetition of the proof of Lemma 4.2 in [8] and is based on (3.3) with $\lambda = 1$ and Q_m and ζ_m , specified below. Introduce

$$Q_m = Q_{r_m} = (0, r_m^2) \times B_{r_m}, \quad m = 1, 2, \dots,$$

where $r_0 = r$ and $r_m = r + (R - r) \sum_{k=1}^m 2^{-k}$. Also let $\zeta_m \in C_0^\infty(\mathbb{R}^{d+1})$ be such that

$$\zeta_m(t, x) = \begin{cases} 1 & \text{on } Q_m \\ 0 & \text{on } \mathbb{R}^{d+1} \setminus [(-r_{m+1}^2, r_{m+1}^2) \times B_{r_{m+1}}] \end{cases}$$

and

$$|(\zeta_m)_x|_0 \leq N \frac{2^{m+1}}{R - r}, \quad |(\zeta_m)_t|_0 \leq N \frac{2^{2m+2}}{(R - r)^2}, \quad |(\zeta_m)_{xx}|_0 \leq N \frac{2^{2m+2}}{(R - r)^2},$$

where N is a constant. In fact, we construct ζ_m as follows. Let $g(t)$ be an infinitely differentiable function defined on \mathbb{R} such that $g(t) = 1$ for $t \leq 1$, $g(t) = 0$, for $t \geq 2$, and $0 \leq g \leq 1$. Then define

$$\begin{aligned} \rho_m(x) &= g(2^{m+1}(R - r)^{-1}(|x| - r_m) + 1), \\ \eta_m(t) &= g(2^{m+1}(R - r)^{-1}(\sqrt{|t|} - r_m) + 1), \\ \zeta_m(t, x) &= \eta_m(t)\rho_m(x). \end{aligned}$$

□

Lemma 4.3. *Let $0 < r < R$ and $\gamma = (\gamma^1, \dots, \gamma^d)$ be a multi-index such that $\gamma^1 = 0, 1, 2$. If h is a sufficiently smooth function defined on Q_R such that $L_0 h = 0$ in Q_R , then*

$$\int_{Q_r} |D_t^m D_x^\gamma h|^2 dx dt \leq N \int_{Q_R} |h|^2 dx dt,$$

where m is a nonnegative integer and $N = N(d, \delta, \gamma, m, R, r)$.

Proof. Since a^{jk} are independent of $t \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$, we have $L_0(D_t^m h) = 0$ and $L_0(D_x^{\gamma'} h) = 0$, where $\gamma' = (0, \gamma^2, \dots, \gamma^d)$. Then the proof is completed using Lemma 4.2 and the argument in the proof of Lemma 4.4 in [8]. □

Throughout the rest of this paper, depending on the context, by $h_{x'}$ we mean one of h_{x^j} , $j = 2, \dots, d$ or the whole collection consisting of them. By h_x we mean one of h_{x^j} , $j = 1, \dots, d$ or the full gradient of h with respect to x . Also, by $h_{xx'}$ we mean one of $h_{x^j x^k}$, where

$j \in \{1, \dots, d\}$ and $k \in \{2, \dots, d\}$ or the collection of them. Norms of these collections are defined arbitrarily.

Lemma 4.4. *Let h be a sufficiently smooth function defined on Q_4 such that $L_0 h = 0$ in Q_4 . Then*

$$\sup_{Q_1} |h_{tt}| + \sup_{Q_1} |h_{tx}| + \sup_{Q_1} |h_{tx'x}| + \sup_{Q_1} |h_{x'xx}| \leq N \|h\|_{L_2(Q_3)},$$

where $N = N(d, \delta)$.

Proof. We prove that

$$\sup_{Q_1} |h| + \sup_{Q_1} |h_{x^1}| \leq N \|h\|_{L_2(Q_r)}, \quad (4.2)$$

where $2 < r < 3$ and $N = N(r, d, \delta)$. If this is true, then using the fact that

$$L_0 h_t = L_0 h_{tt} = L_0 h_{tx'} = L_0 h_{tx'x'} = L_0 h_{x'x'} = L_0 h_{x'x'x'} = 0$$

we obtain

$$\sup_{Q_1} |h_{tt}| + \sup_{Q_1} |h_{tx}| + \sup_{Q_1} |h_{tx'x}| + \sup_{Q_1} |h_{x'x'x}| \leq N \sum_{k+|\gamma| \leq 3} \|D_t^k D_{x'}^\gamma h\|_{L_2(Q_r)}.$$

This and Lemma 4.3 prove all the desired estimates except

$$\sup_{Q_1} |h_{x'x^1x^1}| \leq N \|h\|_{Q_3}.$$

However, this one holds true as well because

$$a^{11} h_{x'x^1x^1} = -h_{x't} - \sum_{j \neq 1 \text{ or } k \neq 1} a^{jk} h_{x'x^jx^k}.$$

To prove (4.2), we observe that, due to the Sobolev embedding theorem, there exist positive constants m and N such that

$$\sup_{Q_1} |h_{x^1}| \leq N \sum_{k+|\gamma| \leq m} (\|D_t^k D_{x'}^\gamma h_{x^1}\|_{L_2(Q_2)} + \|D_t^k D_{x'}^\gamma h_{x^1x^1}\|_{L_2(Q_2)}).$$

By Lemma 4.3, the right side of the above inequality is not greater than a constant $N = N(r, d, \delta)$ times $\|h\|_{L_2(Q_r)}$, $2 < r < 3$. This proves that

$$\sup_{Q_1} |h_{x^1}| \leq N \|h\|_{L_2(Q_r)}.$$

Similarly, we have the same inequality as above with h in place of h_{x^1} . Therefore, (4.2) is proved, so is the lemma. \square

Let $u \in C_0^\infty(\mathbb{R}^{d+1})$. Assume that $a^{jk}(x^1)$ are infinitely differentiable. Then there exists a sufficiently smooth function h defined on Q_4 such that

$$\begin{cases} L_0 h = 0 & \text{in } Q_4 \\ h = u & \text{on } \partial' Q_4 \end{cases}.$$

The functions u and h satisfy the following inequality.

Lemma 4.5. *There exists a constant $N = N(d, \delta)$ such that*

$$\begin{aligned} & \sup_{Q_1} |h_{tt}| + \sup_{Q_1} |h_{tx}| + \sup_{Q_1} |h_{tx'x}| + \sup_{Q_1} |h_{x'xx}| \\ & \leq N \left(\|L_0 u\|_{L_2(Q_4)} + \|u_{xx}\|_{L_2(Q_4)} \right). \end{aligned}$$

Proof. We need only follow the argument in Lemma 4.6 in [8] along with Lemma 4.1 and 4.4. \square

Denote by $(u)_{Q_r(t_0, x_0)}$ the average value of a function u over $Q_r(t_0, x_0)$, that is,

$$(u)_{Q_r(t_0, x_0)} = \int_{Q_r(t_0, x_0)} u(t, x) dx dt.$$

Lemma 4.6. *Let $\kappa \geq 4$ and $r > 0$. Assume that $a^{jk}(x^1)$ are infinitely differentiable. For $u \in C_0^\infty(\mathbb{R}^{d+1})$, we find a smooth function h defined on $Q_{\kappa r}$ such that $L_0 h = 0$ in $Q_{\kappa r}$ and $h = u$ on $\partial' Q_{\kappa r}$. Then there exists a constant $N = N(d, \delta)$ such that*

$$\begin{aligned} & \int_{Q_r} |h_t - (h_t)_{Q_r}|^2 dx dt + \int_{Q_r} |h_{xx'} - (h_{xx'})_{Q_r}|^2 dx dt \\ & \leq N \kappa^{-2} \left[(|L_0 u|^2)_{Q_{\kappa r}} + (|u_{xx}|^2)_{Q_{\kappa r}} \right]. \end{aligned}$$

Proof. By the dilation argument as in the proof of Lemma 4.1, we need to prove our assertion only in the case $r = 1$. In this case, we use Lemma 4.5 and the dilation argument again to obtain

$$\begin{aligned} & \kappa^2 \sup_{Q_{\kappa/4}} |h_{tt}|^2 + \sup_{Q_{\kappa/4}} |h_{tx}|^2 + \kappa^2 \sup_{Q_{\kappa/4}} |h_{tx'x}|^2 + \sup_{Q_{\kappa/4}} |h_{x'xx}|^2 \\ & \leq N \kappa^{-2} \left[(|L_0 u|^2)_{Q_\kappa} + (|u_{xx}|^2)_{Q_\kappa} \right]. \quad (4.3) \end{aligned}$$

Set v to be either h_t or $h_{xx'}$. Then by the fact that $\kappa \geq 4$ it follows that

$$\int_{Q_1} |v - (v)_{Q_1}|^2 dx dt \leq N \sup_{Q_{\kappa/4}} |v_t|^2 + N \sup_{Q_{\kappa/4}} |v_x|^2.$$

This and (4.3) prove the assertion of the lemma in case $r = 1$. The lemma is proved. \square

Lemma 4.7. *There exists a constant $N = N(d, \delta)$ such that, for any $\kappa \geq 4$, $r > 0$, and $u \in C_0^\infty(\mathbb{R}^{d+1})$, we have*

$$\begin{aligned} & \int_{Q_r} |u_t - (u_t)_{Q_r}|^2 dx dt + \int_{Q_r} |u_{xx'} - (u_{xx'})_{Q_r}|^2 dx dt \\ & \leq N\kappa^{d+2} (|L_0 u|^2)_{Q_{\kappa r}} + N\kappa^{-2} (|u_{xx}|^2)_{Q_{\kappa r}}. \end{aligned}$$

Proof. Use Lemma 4.6, 4.2, 4.1, and the argument in the proof of Lemma 4.8 in [8] (also see Remark 4.3 there). \square

5. PROOF OF THEOREM 2.5

We assume in this section that all assumptions of Theorem 2.5 are satisfied. However, in Theorem 5.1 the assumption that $\omega(r) \rightarrow 0$ as $r \downarrow 0$ is not used. Recall that

$$L_0 u = u_t + a^{jk} u_{x^j x^k}.$$

Let \mathbb{Q} be the collection of all $Q_r(t, x)$, $(t, x) \in \mathbb{R}^{d+1}$, $r \in (0, \infty)$. For a function g defined on \mathbb{R}^{d+1} , we denote its (parabolic) maximal and sharp function, respectively, by

$$Mg(t, x) = \sup_{(t, x) \in Q} \int_Q |g(s, y)| dy ds,$$

$$g^\#(t, x) = \sup_{(t, x) \in Q} \int_Q |g(s, y) - (g)_Q| dy ds,$$

where the supremums are taken over all $Q \in \mathbb{Q}$ containing (t, x) .

Theorem 5.1. *Let $\mu, \nu \in (1, \infty)$, $1/\mu + 1/\nu = 1$, and $R \in (0, \infty)$. There exists a constant $N = N(d, \delta, \mu)$ such that, for any $u \in C_0^\infty(Q_R)$, we have*

$$\begin{aligned} & (u_t)^\# + (u_{xx'})^\# \leq N(a_R^\#)^{\alpha/\nu} [M(|u_{xx}|^{2\mu})]^{1/(2\mu)} \\ & + N [M(|L_0 u|^2)]^\alpha [M(|u_{xx}|^2)]^\beta + N [M(|L_0 u|^2)]^{1/2}, \end{aligned} \quad (5.1)$$

where $\alpha = 1/(d+4)$ and $\beta = (d+2)/(2d+8)$.

Proof. Let $\kappa \geq 4$, $r \in (0, \infty)$, and $(t_0, x_0) = (t_0, x_0^1, x_0') \in \mathbb{R}^{d+1}$. Also recall that the sets $\Gamma_r(t, x')$ are introduced in Section 2 and set

$$\begin{aligned} \bar{a}^{jk}(x^1) &= \int_{\Gamma_{\kappa r}(t_0, x_0')} a^{jk}(s, x^1, y') dy' ds \quad \text{if } \kappa r < R, \\ \bar{a}^{jk}(x^1) &= \int_{\Gamma_R} a^{jk}(s, x^1, y') dy' ds \quad \text{if } \kappa r \geq R. \end{aligned}$$

For $\rho > 0$, we denote

$$\mathcal{A}_\rho = (|L_0 u|^2)_{Q_\rho(t_0, x_0)}, \quad \mathcal{B}_\rho = (|u_{xx}|^2)_{Q_\rho(t_0, x_0)}, \quad \mathcal{C}_\rho = (|u_{xx}|^{2\mu})_{Q_\rho(t_0, x_0)}^{1/\mu}.$$

Set $\bar{L}_0 u = u_t + \bar{a}^{jk} u_{x^j x^k}$. Also set w to be either u_t or u_{xx} . Then by Lemma 4.7, we have

$$(|w - (w)_{Q_r(t_0, x_0)}|^2)_{Q_r(t_0, x_0)} \leq N\kappa^{d+2} (|\bar{L}_0 u|^2)_{Q_{\kappa r}(t_0, x_0)} + N\kappa^{-2} \mathcal{B}_{\kappa r}. \quad (5.2)$$

Note that

$$\int_{Q_{\kappa r}(t_0, x_0)} |\bar{L}_0 u|^2 dx dt \leq 2 \int_{Q_{\kappa r}(t_0, x_0)} |L_0 u|^2 dx dt + I, \quad (5.3)$$

where I is a constant times

$$\begin{aligned} \int_{Q_{\kappa r}(t_0, x_0)} |(\bar{L}_0 - L_0)u|^2 dx dt &= \int_{Q_{\kappa r}(t_0, x_0) \cap Q_R} \dots \leq J_1^{1/\nu} J_2^{1/\mu}, \\ J_1 &= \int_{Q_{\kappa r}(t_0, x_0) \cap Q_R} |\bar{a} - a|^{2\nu} dx dt, \\ J_2 &= \int_{Q_{\kappa r}(t_0, x_0) \cap Q_R} |u_{xx}|^{2\mu} dx dt \leq N(\kappa r)^{d+2} (\mathcal{C}_{\kappa r})^\mu. \end{aligned}$$

Observe that if $\kappa r < R$,

$$\begin{aligned} J_1 &\leq N \int_{x_0^1 - \kappa r}^{x_0^1 + \kappa r} \int_{\Gamma_{\kappa r}(t_0, x_0^1)} |\bar{a} - a| dx' dt dx^1 \\ &\leq N(\kappa r)^{d+2} a_{\kappa r}^\# \leq N(\kappa r)^{d+2} a_R^\#. \end{aligned}$$

In case $\kappa r \geq R$,

$$J_1 \leq N \int_{-R}^R \int_{\Gamma_R} |\bar{a} - a| dx' dt dx^1 \leq NR^{d+2} a_R^\# \leq N(\kappa r)^{d+2} a_R^\#.$$

From (5.3) and the above estimates, we have

$$(|\bar{L}_0 u|^2)_{Q_{\kappa r}(t_0, x_0)} \leq N(a_R^\#)^{1/\nu} \mathcal{C}_{\kappa r} + N\mathcal{A}_{\kappa r}.$$

This, together with (5.2), gives us

$$\begin{aligned} (|w - (w)_{Q_r(t_0, x_0)}|^2)_{Q_r(t_0, x_0)} &\leq N\kappa^{d+2} (a_R^\#)^{1/\nu} \mathcal{C}_{\kappa r} \\ &\quad + N\kappa^{d+2} \mathcal{A}_{\kappa r} + N\kappa^{-2} \mathcal{B}_{\kappa r}. \end{aligned} \quad (5.4)$$

Now observe that $\mathcal{B}_{\kappa r} \leq M(|u_{xx}|^2)(t, x)$ for any $(t, x) \in Q_r(t_0, x_0)$. Similar inequalities hold true for $\mathcal{A}_{\kappa r}$ and $\mathcal{C}_{\kappa r}$. From this fact and (5.4) it follows that, for any $(t, x) \in \mathbb{R}^{d+1}$ and $Q \in \mathbb{Q}$ such that $(t, x) \in Q$,

$$(|w - (w)_Q|^2)_Q \leq N \left(\kappa^{d+2} (a_R^\#)^{1/\nu} \mathcal{C}(t, x) + \kappa^{d+2} \mathcal{A}(t, x) + \kappa^{-2} \mathcal{B}(t, x) \right),$$

where

$$\mathcal{A} = M(|L_0 u|^2), \quad \mathcal{B} = M(|u_{xx}|^2), \quad \mathcal{C} = (M(|u_{xx}|^{2\mu}))^{1/\mu}.$$

Note that the above inequality is proved for $\kappa \geq 4$. In case $0 < \kappa < 4$, we have

$$\begin{aligned} \int_Q |w - (w)_Q|^2 dx dt &\leq (|w|^2)_Q \leq N (|L_0 u|^2)_Q + N (|u_{xx}|^2)_Q \\ &\leq N (\mathcal{A}(t, x) + 16\kappa^{-2}\mathcal{B}(t, x)) \end{aligned}$$

for $(t, x) \in Q \in \mathbb{Q}$. Therefore, we finally have

$$\begin{aligned} (|w - (w)_Q|^2)_Q &\leq N\kappa^{d+2}(a_R^\#)^{1/\nu}\mathcal{C}(t, x) \\ &\quad + N(\kappa^{d+2} + 1)\mathcal{A}(t, x) + N\kappa^{-2}\mathcal{B}(t, x) \end{aligned}$$

for all $\kappa > 0$, $(t, x) \in \mathbb{R}^{d+1}$, and $Q \in \mathbb{Q}$ satisfying $(t, x) \in Q$.

Take the supremum of the left side of the above inequality over all $Q \in \mathbb{Q}$ containing (t, x) , and then minimize the right-hand side with respect to $\kappa > 0$. Also observe that

$$\left(\int_Q |w - (w)_Q| dx dt \right)^2 \leq \int_Q |w - (w)_Q|^2 dx dt$$

Then we obtain

$$[w^\#]^2(t, x) \leq N\mathcal{A}(t, x) + [(a_R^\#)^{1/\nu}\mathcal{C} + \mathcal{A}]^{2/(d+4)}\mathcal{B}^{(d+2)/(d+4)}(t, x).$$

Here $\mathcal{B} \leq \mathcal{C}$ and this leads to (5.1). \square

Corollary 5.2. *For $p > 2$, there exist constants $R = R(d, \delta, p, \omega)$ and $N = N(d, \delta, p)$ such that, for any $u \in C_0^\infty(Q_R)$, we have*

$$\|u_{xx}\|_{L_p(\mathbb{R}^{d+1})} \leq N\|L_0 u\|_{L_p(\mathbb{R}^{d+1})}.$$

Proof. Set $L_p = L_p(\mathbb{R}^{d+1})$. Choose a number μ such that $p > 2\mu > 1$. Then we use (5.1) together with the Fefferman-Stein theorem on sharp functions, Hölder's inequality, and the Hardy-Littlewood maximal function theorem to obtain

$$\|u_t\|_{L_p} + \|u_{xx'}\|_{L_p} \leq N(a_R^\#)^{\alpha/\nu}\|u_{xx}\|_{L_p} + N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta} + N\|L_0 u\|_{L_p}, \quad (5.5)$$

where, as noted in Theorem 5.1, $1/\mu + 1/\nu = 1$ and $2\alpha + 2\beta = 1$. Now we notice that

$$u_{x^1 x^1} = \frac{1}{a^{11}} \left(L_0 u - u_t - \sum_{j \neq 1, k \neq 1} a^{jk} u_{x^j x^k} \right).$$

From this and (5.5), we have

$$\|u_{xx}\|_{L_p} \leq N(a_R^\#)^{\alpha/\nu}\|u_{xx}\|_{L_p} + N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta} + N\|L_0 u\|_{L_p}.$$

Choose an appropriate R such that

$$N(a_R^\#)^{\alpha/\nu} \leq 1/2.$$

Then

$$\frac{1}{2}\|u_{xx}\|_{L_p} \leq N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta} + N\|L_0 u\|_{L_p}.$$

This finishes the proof. \square

Lemma 5.3. *Let $T \in (0, \infty]$. Then there exists $\lambda_0 = \lambda_0(d, \delta, K, p, \omega) \geq 0$ such that, for all $\lambda \geq \lambda_0$ and $u \in \dot{W}_p^{1,2}(\Omega_T)$,*

$$\lambda\|u\|_{L_p(\Omega_T)} + \|u_{xx}\|_{L_p(\Omega_T)} + \|u_t\|_{L_p(\Omega_T)} \leq N\|Lu - \lambda u\|_{L_p(\Omega_T)},$$

where $N = N(d, \delta, K, p, \omega)$ (independent of T).

Proof. We have an L_p -estimate for functions with small compact support. Thus the rest of the proof can be done by following the argument in [9]. \square

Now Theorem 2.5 follows from the above lemma and the argument in [9]. This ends the proof of Theorem 2.5.

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127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: dykim@math.umn.edu

127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: krylov@math.umn.edu